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Semi-linear Poisson-mediated Flocking in a Cucker-Smale Model ⋆

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Abstract: We propose a family of compactly supported parametric interaction functions in the general Cucker-Smale flocking dynamics such that the mean-field macroscopic system of mass and momentum balance equations with non-local damping terms can be converted from a system of partial integro-differential equations to an augmented system of partial differential equations in a compact set. We treat the interaction functions as Green's functions for an operator corresponding to a semi-linear Poisson equation and compute the density and momentum in a translating reference frame, i.e. one that is taken in reference to the flock's centroid. This allows us to consider the dynamics in a fixed, flock-centered compact set without loss of generality. We approach the computation of the non-local damping using the standard finite difference treatment of the chosen differential operator, resulting in a tridiagonal system which can be solved quickly.

Keywords: Control of Distributed Parameter Systems, Networked Control Systems, Large Scale Systems

1. INTRODUCTION

Collective motion of autonomous agents is a widespread phenomenon appearing in numerous applications ranging from animal herding to complex networks and social dynamics (Okubo, 1986; Cucker and Smale, 2007; Giardina, 2008).

In general, there are two broad approaches when investigating the underlying dynamics for flocks or swarms: the microscopic, particle models described by ordinary differential equations (ODEs) or stochastic differential equations, and the macroscopic continuum models, described by partial differential equations (PDEs). Agent-based models assume behavioral rules at the individual level, such as velocity alignment, attraction, and repulsion (Cucker and Smale, 2007; Giardina, 2008; Ballerini et al., 2008) and are often used in numerical simulations and in learning schemes where the interaction rules are inferred (Matei et al., 2019). As the number of interacting agents gets large, the agent-based models become computationally expensive (Carrillo et al., 2010). Considering pairwise interactions, the growth is $O(N^2)$, where N is the number of agents. As we approach the mean-field limit, it is useful to consider the probability density of the agents. Using Vlasov-like arguments (Carrillo et al., 2010), we can construct an equation analogous to the Fokker-Planck-Kolmogorov equation. We can then define momentum and density and construct a system of compressible hydrodynamic PDEs (Carrillo et al., 2010; Shvydkoy and Tadmor, 2017).

In flocking dynamics (Cucker and Smale, 2007; Carrillo et al., 2010), the velocity alignment term is not only nonlocal but can also be nonlinear (Shvydkoy and Tadmor, 2017; Mao et al., 2018). The computation of the corresponding hydrodynamic equations with nonlocal forces becomes quite costly due to the approximation of the convolution integrals or integral transforms using the various quadrature methods. The simplest 'quadrature' method is the Riemann sum, whose complexity is $O(n^2)$, where n is the number of grid points, when estimating a convolution integral as a convolution sum in one dimension. On the other hand, an equivalent solution may be obtained using finite differences if the interaction kernel is associated with a differential operator. If that operator can be put into a sparse form, ideally a tridiagonal form, a solution can be obtained efficiently.

In this work, we modify the classical Cucker-Smale model of nonlocal particle interaction for velocity consensus (Cucker and Smale, 2007; Ha et al., 2009). We propose a family of parametric interaction functions in $\mathbb{R}^d$, $d \in \{1,2,3\}$, that are Green's functions for appropriately defined linear partial differential operators, which allow us to speed-up computation of the nonlocal interaction terms. We investigate the conditions under which time-asymptotic flocking is achieved in the microscopic formulation in a centroid-fixed frame. We solve the macroscopic formulation using the Kurganov-Tadmor MUSCL finite volume method (Kurganov and Tadmor, 2000) and a second-order finite difference discretization of our chosen differential operator. The method is compared to bulk variables computed from the microscopic formulation for validation.

The rest of the manuscript is organized as follows: Section 2 introduces the agent-based Cucker-Smale flocking dynamics and the macroscopic Euler equations. Section 3 describes the conversion of the Euler equations to an augmented system of PDEs, and the formulation of the boundary value problem. In Section 4 a family of interaction functions is proposed and the computation process is explained. Finally, Section 5 compares the numerical results and Section 6 concludes the paper.

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2. MATHEMATICAL MODELS

In this section we introduce the Cucker-Smale dynamics under general interaction functions, define time-asymptotic flocking, and present the mean-field macroscopic equations.

2.1 The Cucker-Smale Model

Consider an interacting system of N identical autonomous agents with unit mass in $\mathbb{R}^d$, $d \in \{1,2,3\}$. Let $x_i(t),\ v_i(t) \in \mathbb{R}^d$ represent the position and velocity of the i^{th}-particle at each time $t \geq 0$, respectively, for $1 \leq i \leq N$. Then the general Cucker-Smale dynamical system (Cucker and Smale, 2007) of $(2N)$ ODEs reads as:

$$\begin{cases} \dfrac{dx_i}{dt} &= v_i \\ \dfrac{dv_i}{dt} &= \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \psi(x_j,x_i)(v_j - v_i) \end{cases} \tag{1}$$

where $x_i(0)$, are $v_i(0)$ are given for all $i = 1,\dots,N$, and $\psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ represents the interaction function between each pair of particles.

The center of mass system (x_c, v_c) of $\{(x_i,v_i)\}_{i=1}^N$ is defined as

$$x_c = \frac{1}{N}\sum_{i=1}^{N} x_i, \quad v_c = \frac{1}{N}\sum_{i=1}^{N} v_i \tag{2}$$

When ψ is symmetric, i.e., $\psi(x,s) = \psi(s,x)$, system (1) implies

$$\frac{dx_c}{dt} = v_c, \quad \frac{dv_c}{dt} = 0 \tag{3}$$

which gives the explicit solution

$$x_c(t) = x_c(0) + t v_c(0),\ t \geq 0 \tag{4}$$

2.2 Asymptotic Flocking

We investigate the additional assumptions on the initial conditions and the interaction function ψ, such that system (1) converges to a velocity consensus, a phenomenon known in the literature as *time-asymptotic flocking*, defined in terms of the center of mass system as

Definition 1 (Asymptotic Flocking)**.** *An $N-$body interacting system $\mathcal{G} = \{(x_i,v_i)\}_{i=1}^N$ exhibits time-asymptotic flocking if and only if the following two relations hold:*

- *(Velocity alignment):* $\lim_{t\to\infty} \sum_{i=1}^N \|v_i(t) - v_c(t)\|^2 = 0$,
- *(Spatial coherence):* $\sup_{0\leq t\leq\infty} \sum_{i=1}^N \|x_i(t) - x_c(t)\|^2 < \infty$.

We consider the new variables

$$(\hat{x}_i, \hat{v}_i) := (x_i - x_c, v_i - v_c) \tag{5}$$

which correspond to the fluctuations around the center of mass system, and define $\hat{x} := (\hat{x}_1,\dots,\hat{x}_N)$, $\hat{v} := (\hat{v}_1,\dots,\hat{v}_N)$, $|\hat{x}| = \left(\sum_{i=1}^N \|\hat{x}_i\|^2\right)^{1/2}$, and $|\hat{v}| = \left(\sum_{i=1}^N \|\hat{v}_i\|^2\right)^{1/2}$, where $\|\cdot\|$ represents the standard l_2-norm in $\mathbb{R}^d$. Based on Definition 1, asymptotic flocking is achieved if

$$|\hat{x}(t)| < \infty, t \geq 0, \text{ and } \lim_{t\to\infty} |\hat{v}(t)| = 0 \tag{6}$$

We first notice that

$$\frac{d|\hat{x}|^2}{dt} = 2\left\langle \frac{d\hat{x}}{dt}, \hat{x} \right\rangle \leq 2|\hat{x}||\hat{v}| \tag{7}$$

which implies

$$\frac{d|\hat{x}|}{dt} \leq |\hat{v}| \tag{8}$$

Suppose the interaction function ψ is chosen such that $\psi(x,s) = \tilde{\psi}(\|x-s\|)$, with $\tilde{\psi} : \mathbb{R}_+ \to \mathbb{R}_+$ being a non-negative and non-increasing function. Then $(\hat{x}_i, \hat{v}_i)$ are governed by the dynamical system (1), and

$$\begin{aligned} \frac{d|\hat{v}|^2}{dt} &= -\frac{1}{N}\sum_{1\leq i,j\leq N} \tilde{\psi}(\|\hat{x}_j - \hat{x}_i\|)\|\hat{v}_j - \hat{v}_i\|^2 \\ &\leq -\frac{1}{N}\tilde{\psi}(2|\hat{x}|)\sum_{1\leq i,j\leq N} \|\hat{v}_j - \hat{v}_i\|^2 \\ &= -\frac{2}{N}\tilde{\psi}(2|\hat{x}|)|\hat{v}|^2 \end{aligned} \tag{9}$$

which implies

$$\frac{d|\hat{v}|}{dt} \leq -\frac{2}{N}\tilde{\psi}(2|\hat{x}|)|\hat{v}| := -\phi(|\hat{x}|)|\hat{v}| \tag{10}$$

where we have used the fact that $\sum_{i=1}^N \hat{v}_i(t) = 0$, $t \geq 0$, and

$$\max_{1\leq i,j\leq N} \|\hat{x}_i - \hat{x}_j\| \leq 2|\hat{x}| \tag{11}$$

The following Theorem by (Ha et al., 2009) provides sufficient conditions for time-asymptotic flocking.

Theorem 1. *Suppose $(|x|,|v|)$ satisfy the system of dissipative differential inequalities (8), (10) with $\phi \geq 0$. Then if $|v(0)| < \int_{|x(0)|}^{\infty} \phi(s)ds$, there is a $x_M \geq 0$ such that $|v(0)| = \int_{|x(0)|}^{x_M} \phi(s)ds$, and for every $t \geq 0$, $|x(t)| \leq x_M$, and $|v(t)| \leq |v(0)|e^{-\phi(x_M)t}$.*

The following is an immediate consequence of Theorem 1.

Proposition 1. *Let $\mathcal{G} = \{(x_i,v_i)\}_{i=1}^N$ be an $N-$body interacting system with dynamics given by (1). Suppose $\psi(x,s) = \tilde{\psi}(\|x-s\|)$, with $\tilde{\psi} : \mathbb{R}_+ \to \mathbb{R}_+$ being a non-negative and non-increasing function. Then if $|v(0) - v_c(0)| < \int_{|x(0)-x_c(0)|}^{\infty} \frac{2}{N}\tilde{\psi}(2s)ds$, $\mathcal{G}$ exhibits time-asymptotic flocking.*

2.3 The Mean-Field Limit

Consider the empirical joint probability distribution of the particle positions and velocities $\{x_i, v_i\}_{i=1}^N$

$$F_{xv}^N(t,x,v) := \frac{1}{N}\sum_{i=1}^{N} \delta(x_i, v_i) \tag{12}$$

where $\delta(\cdot,\cdot)$ is the Dirac measure on $\mathbb{R}^{2d}$. As the number of particles $N \to \infty$, we can use McKean-Vlasov arguments to show that the empirical distribution converges weakly to a distribution whose density f_{xv} evolves according to the forward Kolmogorov equation (Carrillo et al., 2010)

$$\begin{aligned} &\partial_t f_{xv} + \nabla_x \cdot (v f_{xv}) + \nabla_v \cdot (A f_{xv}) = 0 \\ &A := \int_{\mathbb{R}^{2d}} \psi(x,s)(w - v) f_{xv}(t,s,w) ds dw. \end{aligned} \tag{13}$$

We define

$$\begin{aligned} \rho(t,x) &:= \int_{\mathbb{R}^d} f_{xv}(t,x,v)dv \\ m(t,x) &:= \rho(t,x)u(t,x) := \int_{\mathbb{R}^d} v f_{xv}(t,x,v)dv. \end{aligned} \tag{14}$$

which are the marginal probability and momentum density functions. Substituting these into (13) yields the following $(d+1)$ compressible Euler equations with non-local forcing:

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0 \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) = \rho \mathcal{L}_\psi(\rho u) - \rho u \mathcal{L}_\psi \rho \end{cases} \tag{15}$$

where u is the mean velocity, $\rho(0,x)$ and $u(0,x)$ are given and

$$\mathcal{L}_\psi f(t,x) := \int_{\mathbb{R}^d} \psi(x,s) f(t,s) ds. \tag{16}$$

3. SEMI-LINEAR POISSON MEDIATED FLOCKING

3.1 Conversion to a system of PDEs

We think of the function ψ as a Green's function, i.e., as the impulse response of a linear differential equation, represented by the operator $\mathcal{L}_x$, such that

$$\mathcal{L}_x y(t,x) = g(t,x) \tag{17}$$

implies

$$y(t,x) = \int_{\mathbb{R}^d} \psi(x,s) g(t,s) ds \tag{18}$$

which results in

$$\mathcal{L}_\psi^{-1} = \mathcal{L}_x \tag{19}$$

for all $t \geq 0$, where

$$\mathcal{L}_x \psi(x,s) = \delta(x-s),\ x,s \in \mathbb{R}^d. \tag{20}$$

Then the following proposition holds:

Proposition 2. *Suppose ψ is a Green's function with respect to a linear differential operator $\mathcal{L}_x$. Then system (15) is equivalent to the augmented system of $(2d+2)$ partial differential equations:*

$$\begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0 \\ \mathcal{L}_x y = \begin{bmatrix} \rho u & \rho \end{bmatrix}^T \\ \partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) = \sum_{i=1}^{d} (\rho y_i - \rho u_i y_{d+1}) \cdot \hat{e}_i \end{cases} \tag{21}$$

where $\{\hat{e}_i\}_{i=1}^d$ is the standard basis in $\mathbb{R}^d$.

3.2 The Boundary Value Problem

Due to the time-dependence of the center of mass (4), x_i, $i = 1,\dots,N$, will escape any fixed and open bounded domain $\Omega \subset \mathbb{R}^d$, unless in the trivial case where $v_c(0) = 0$. Because of the flocking behavior (Definition 1), the position fluctuations with respect to the center of mass are uniformly bounded, i.e.,

$$\sup_{0 \leq t \leq \infty} \sum_{i=1}^{N} \|x_i(t) - x_c(t)\|^2 < \infty \tag{22}$$

and, therefore we can define a Boundary Value Problem (BVP) in the moving domain

$$\Omega_c(t) = \{x + x_c(t) : x \in \Omega\} \tag{23}$$

where it is assumed that $0_d \in \Omega$, 0_d being the origin of $\mathbb{R}^d$.

We notice that solving system (21) for (x,u), $x \in \Omega_c$ is equivalent to solving it for the fluctuation variables $(\hat{x},\hat{u})$ (5), with $\hat{x} \in \Omega$.

We note that the boundedness of the domain has an effect on both the Green's function and the flocking behavior of the system of interacting particles, which should satisfy

$$x_i(t) - x_c(t) \in \Omega,\ i = 1,\dots,N,\ t \geq 0. \tag{24}$$

4. ONE-DIMENSIONAL CASE

The BVP of the augmented system of PDEs (21) for $d = 1$, on $\Omega = \{\hat{x} \in [-\frac{L}{2}, \frac{L}{2}]\}$ reads as:

$$\begin{cases} \partial_t \rho + \partial_{\hat{x}} (\rho u) = 0 \\ \mathcal{L}_{\hat{x}} y = \begin{bmatrix} \rho u & \rho \end{bmatrix}^T \\ \partial_t (\rho u) + \partial_{\hat{x}} (\rho u^2) = \rho y_1 - \rho u y_2 \end{cases} \tag{25}$$

with homogeneous Dirichlet boundary conditions and initial conditions

$$\rho(0,\hat{x}) = \rho_0(\hat{x}), \quad u(0,\hat{x}) = u_0(\hat{x}) \tag{26}$$

which are smooth functions.

We select the linear partial differential operator

$$\mathcal{L}_x = -\frac{1}{2k}\left(\frac{\partial^2}{\partial x^2} - \lambda^2\right) \tag{27}$$

with $k \neq 0$ and $\lambda \neq 0$, for which the associated parametric family of Green's functions with homogeneous Dirichlet boundary conditions on $[0,L]$ reads as:

$$\hat{\psi}(x,s) = \begin{cases} c_1(s)(e^{\lambda x} - e^{-\lambda x}) & s \leq x \\ c_2(s)(e^{\lambda(x-2L)} - e^{-\lambda x}) & s > x \end{cases} \tag{28}$$

$$\begin{aligned} c_1(s) &= \frac{k}{\lambda(e^{-2L\lambda} - 1)}(e^{\lambda(s-2L)} - e^{-\lambda s}) \\ c_2(s) &= \frac{k}{\lambda(e^{-2L\lambda} - 1)}(e^{\lambda s} - e^{-\lambda s}) \end{aligned} \tag{29}$$

The solution over any interval of length L can be obtained by a simple translation of coordinates.

The profile of the Green's function $\hat{\psi}$ and the effect of the bounded domain on on it is illustrated in Fig. 1, where, for different fixed values of x, $\hat{\psi}(x,s)$ is compared to the function

$$\psi(x,s) = \frac{k}{\lambda} e^{-\lambda \|x-s\|} \tag{30}$$

which is the Green's function corresponding to $\mathcal{L}_x$ in an infinite domain. We note that the parameters (k,λ,L) generate a family of interaction functions (see also (Mavridis et al., 2020)) that can simulate widely used interaction functions as the one found in the original Cucker-Smale model (Cucker and Smale, 2007):

$$G(x,s) = \frac{K}{(1 + \|x-s\|^2)^\gamma} \tag{31}$$

for given parameters (K,γ).

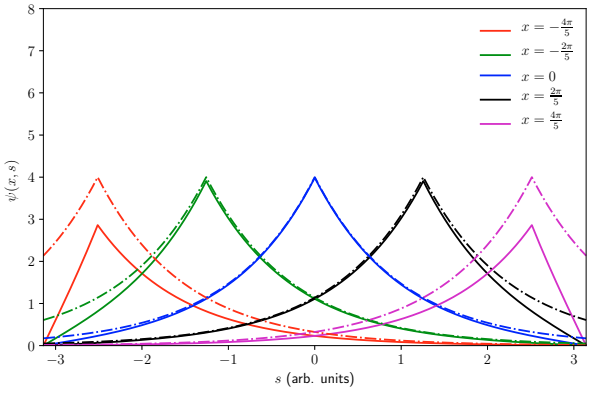

Fig. 1. Illustration of $\hat{\psi}(x,\cdot)$ (28) for different values of x, and for $\lambda = 1$, $k = 4$ on $[-\pi,\pi]$. The function $\psi(x,s) = \frac{k}{\lambda} e^{-\lambda\|x-s\|}$, which is the Green's function for $\mathcal{L}_x$ in infinite domain, is depicted in the dashed-dotted lines.

4.1 Asymptotic Flocking

Next we provide sufficient conditions such that the solution $\{(x_i(t), v_i(t))\}_{i=1}^N$, $t \geq 0$, of system (1) with interaction function $\hat{\psi}$ as defined in (28), (29), satisfy the flocking conditions in Definition 1, with $\hat{x}_i(t) \in \Omega$, for all $t \geq 0$.

Similar to Section 2.2, we notice that

$$\frac{d|\hat{x}|}{dt} \leq |\hat{v}| \tag{32}$$

From (11) and the fact that $\|\hat{x}_i\| \leq \max_{1\leq i,j\leq N}\|\hat{x}_i - \hat{x}_j\|$, we get

$$|\hat{x}| \leq \frac{\hat{x}_M}{2} \implies \|\hat{x}_i\| \leq \hat{x}_M,\ i = 1,\dots,N \tag{33}$$

Therefore, we are interested in showing asymptotic flocking with $|\hat{x}(t)| \in [0, \frac{\hat{x}_M}{2}]$, for all $t \geq 0$.

For any given initial conditions $|\hat{x}_i(0)|$, there is a large enough value of L such that there exist an $\hat{x}_M \in [0, \frac{L}{2})$ for which

$$\hat{x}_M > 2|\hat{x}(0)| \tag{34}$$

From (28), (29) it follows that for $|\hat{x}| \leq \frac{\hat{x}_M}{2}$,

$$\begin{aligned}\hat{\psi}(x_j, x_i) &\geq \hat{\psi}(-\hat{x}_M, \|\hat{x}_j - \hat{x}_i\|) \\ &\geq \hat{\psi}(-\hat{x}_M, 2|\hat{x}|)\end{aligned} \tag{35}$$

which implies that

$$\begin{aligned}\frac{d|\hat{v}|^2}{dt} &= -\frac{1}{N}\sum_{1\leq i,j\leq N}\hat{\psi}(\hat{x}_j, \hat{x}_i)\|\hat{v}_j - \hat{v}_i\|^2 \\ &\leq -\frac{2}{N}\hat{\psi}(-\hat{x}_M, 2|\hat{x}|)|\hat{v}|^2\end{aligned} \tag{36}$$

and

$$\frac{d|\hat{v}|}{dt} \leq -\frac{2}{N}\hat{\psi}(-\hat{x}_M, 2|\hat{x}|)|\hat{v}| := -\phi(|\hat{x}|)|\hat{v}| \tag{37}$$

Next we notice that the Lyapunov function

$$V(|x|, |v|) := |\hat{v}| + \int_{\alpha}^{|\hat{x}|}\phi(s)ds,\ \alpha \geq 0 \tag{38}$$

is non-increasing along the solutions of $(|\hat{x}(t)|, |\hat{v}(t)|)$ of the system of dissipative differential inequalities (8) and (10), for $|\hat{x}(t)| \leq \frac{\hat{x}_M}{2}$, since

$$\begin{aligned}\frac{d}{dt}V(|\hat{x}|, |\hat{v}|) &= \frac{d|\hat{v}|}{dt} + \phi(|\hat{x}|)\frac{d|\hat{x}|}{dt} \\ &\leq \phi(|\hat{x}|)\left(-|v| + \frac{d|\hat{x}|}{dt}\right) \\ &\leq 0\end{aligned} \tag{39}$$

which implies that

$$|\hat{v}(t)| + \int_{|\hat{x}_0|}^{|\hat{x}|}\phi(s)ds \leq |\hat{v}(0)|,\ |\hat{x}| \leq \frac{\hat{x}_M}{2} \tag{40}$$

Choosing the initial velocity $|\hat{v}(0)|$ such that $|\hat{v}(0)| < \int_{|\hat{x}(0)|}^{\hat{x}_M/2}\phi(s)ds$, and, since ϕ is non-negative for $|\hat{x}(t)| \leq \frac{\hat{x}_M}{2}$, there exists a $\bar{x} \in [|\hat{x}(0)|, \frac{\hat{x}_M}{2}]$ for which

$$|\hat{v}(0)| = \int_{|\hat{x}(0)|}^{\bar{x}}\phi(s)ds \tag{41}$$

Suppose there exists a $t^* \geq 0$, such that $\hat{x}^* := |\hat{x}(t^*)| \in (\bar{x}, \frac{\hat{x}_M}{2}]$. Then

$$\int_{|\hat{x}(0)|}^{\hat{x}^*}\phi(s)ds > |v(0)| \tag{42}$$

which contradicts (40). Therefore

$$|\hat{x}(t)| \leq \bar{x} \leq \frac{\hat{x}_M}{2},\ t \geq 0 \tag{43}$$

and from (10) and the Grönwall-Bellman inequality

$$|\hat{v}(t)| \leq |\hat{v}(0)|e^{-\phi(\bar{x})t},\ t \geq 0. \tag{44}$$

4.2 Conservation of Mass and Momentum

Lemma 1. *The operator (27) $\mathcal{L}_x$ on $C^{\infty}_{\mathbb{R},C}(\Omega)$, the space of compactly supported test functions, is self-adjoint and invertible, and therefore has a self-adjoint inverse $\mathcal{L}_x^{-1}$ on $C^{\infty}_{\mathbb{R},C}(\Omega)$.*

Proof. Self-adjointness of the inverse follows immediately from self-adjointness of $\mathcal{L}_x$ and the existence of the inverse (Taylor, 2010). It is clear that $\mathcal{L}_x$ has an inverse since the Green's function is nontrivial.

We shall now show that the operator $\mathcal{L}_x$ is self-adjoint on $C^{\infty}_{\mathbb{R},C}(\Omega)$. Consider two functions $u, w \in C^{\infty}_{\mathbb{R},C}(\Omega)$, $u \neq w$, the space of test functions, and associated $f_u, f_w \in C^{\infty}_{\mathbb{R},C}$, $f_u := \mathcal{L}_x u, f_w := \mathcal{L}_x w$. Let $\Omega := [-\frac{L}{2}, \frac{L}{2}]$. We have

$$\int_{\Omega}(w\mathcal{L}_x u - u\mathcal{L}_x w)dx = -\frac{1}{2k}\int_{\Omega}(w\partial_x^2 u - u\partial_x^2 w)dx. \tag{45}$$

since the semi-linear term drops out. Using Green's second identity, and the compact support of u, w, we have that

$$\int_{\Omega}(w\partial_x^2 u - u\partial_x^2 w)dx = \int_{\partial\Omega}(w\partial_{\mathbf{n}}u - u\partial_{\mathbf{n}}w)dx = 0. \tag{46}$$

Thus, $\mathcal{L}_x$ is self-adjoint and has a self-adjoint inverse, i.e.

$$\int_{\Omega}(f_w\mathcal{L}_x^{-1}f_u - f_u\mathcal{L}_x^{-1}f_w)dx = \int_{\Omega}(f_w u - f_u w)dx = 0. \tag{47}$$

□

Proposition 3. *If y is compactly supported, and psi is as given, then mass and momentum are conserved, i.e.*

$$\frac{d}{dt}\int_{\Omega}\begin{bmatrix}\rho & \rho u\end{bmatrix}^T d\hat{x} = \int_{\Omega}\begin{bmatrix}0 & \rho y_1 - \rho u y_2\end{bmatrix}^T d\hat{x} = 0. \tag{48}$$

Proof. We obtain (48) by simply integrating the conservation laws in (25) over the entire space and apply the Leibniz rule. The conclusion follows directly from the self adjointness of the inverse in (47). The proposition holds for any self-adjoint alignment operator. □

4.3 Computational Methods

For compactness, we re-write the PDEs (25) as

$$\begin{cases}\partial_t U + \partial_{\hat{x}}F(U) = S(U,Y) \\ \mathcal{L}_{\hat{x}}Y = U\end{cases} \tag{49}$$

with $U = [\rho, \rho u]^T$, $Y = [y_2, y_1]^T$, $F = \begin{bmatrix}\rho u, \rho u^2\end{bmatrix}^T$, and $S = [0, \rho y_1 - \rho u y_2]^T$. Recall the transformation $m = \rho u$. From this, the flux Jacobian is given by

$$\mathbf{D}_U F := \begin{bmatrix}0 & 1 \\ -u^2 & 2u\end{bmatrix} \tag{50}$$

which is not diagonalizable, and thus the system is only weakly hyperbolic. Its eigenvalues are $\pm u$. With these notations established, we now detail the numerical solution of the PDEs.

Hyperbolic Solver. To solve the hyperbolic system, we apply the finite volume method (LeVeque, 2002). To begin, we define the sequence of points $\{\hat{x}_0, \dots, \hat{x}_i, \dots, \hat{x}_N\}$ which are the centers of the cells $I_i := [\hat{x}_{i-\frac{1}{2}}, \hat{x}_{i+\frac{1}{2}})$. Then, we average the PDE over these cells, which gives

$$\frac{1}{\lambda(I_i)}\frac{d}{dt}\int_{I_i}U d\hat{x} = -\frac{1}{\lambda(I_i)}\int_{I_i}\partial_{\hat{x}}F d\hat{x} + \frac{1}{\lambda(I_i)}\int_{I_i}S d\hat{x} \tag{51}$$

where $\lambda(\cdot)$ denotes the length of an interval. Suppose these are identical, so $\Delta\hat{x} := \lambda(I_i)\forall i$. Then, using the divergence theorem, and replacing the integrals of U, F, S with their cell-averages, i.e. their midpoint values $\bar{U}, \bar{F}, \bar{S}$, we obtain

$$\frac{d}{dt}\bar{U}_i = -\frac{1}{\Delta\hat{x}}(\bar{F}_{i+\frac{1}{2}} - \bar{F}_{i-\frac{1}{2}}) + \bar{S}_i \tag{52}$$

where $\bar{U}_i := \bar{Y}(\hat{x}_i), \bar{F}_i := \bar{F}(\hat{x}_i), \bar{S} := \bar{S}(\hat{x}_i)$. In this work, we employ the second-order strong stability preserving Runge-Kutta scheme (Kurganov and Tadmor, 2000) for time integration. For the fluxes, we assume piecewise linearity and use the Kurganov-Tadmor flux (Kurganov and Tadmor, 2000). The fluxes are given by

$$\begin{aligned} \bar{F}_{i+\frac{1}{2}} &:= \frac{1}{2}[F^*_i + F^*_{i+1} - \max\{|u^*_i|, |u^*_{i+1}|\}(U^*_{i+1} - U^*_i)] \\ U^*_{i+1} &:= U_{i+1} - \frac{\Delta\hat{x}}{2} minmod(\frac{U_{i+2} - U_{i+1}}{\Delta\hat{x}}, \frac{U_{i+1} - U_i}{\Delta\hat{x}}) \\ U^*_i &:= U_i + \frac{\Delta\hat{x}}{2} minmod(\frac{U_{i+1} - U_i}{\Delta\hat{x}}, \frac{U_i - U_{i-1}}{\Delta\hat{x}}) \end{aligned} \tag{53}$$

where $minmod(a,b) := \frac{1}{2}(sign(a) + sign(b)) \min(|a|, |b|)$.

Elliptic Solver. To solve the elliptic equations, we apply the classical second-order finite difference method, which is

$$\frac{y^j_{i+1} - 2y^j_i + y^j_{i-1}}{\Delta\hat{x}^2} - \lambda^2 y^j_i = -2kU^j_i \tag{54}$$

Over the interior points, this yields linear equations

$$(\frac{1}{\Delta\hat{x}^2}\mathbf{A} - \lambda^2\mathbf{I})y^j_{int} = -2kU^j_{int} - \frac{1}{\Delta\hat{x}^2}\begin{bmatrix} y^j_0 & 0 & \dots & 0 & y^j_N \end{bmatrix}^T, \tag{55}$$

$$\mathbf{A} = \begin{bmatrix} -2 & 1 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix} \tag{56}$$

The matrix in (55) is tridiagonal, so banded matrix algorithms (Golub and Van Loan, 2013) can be used to solve the corresponding system of equations. As shown in Fig. 2, using finite differences is much faster than a convolution (Riemann) sum, even when the embarrassing parallelism of the sum is exploited.

Particle Solver. We solve the system of particle equations using the velocity Verlet algorithm (Mao et al., 2018). Given a system of ODEs of the form

$$\begin{cases} \dfrac{dx}{dt} &= v \\ \dfrac{dv}{dt} &= a(x,v,t), \end{cases} \tag{57}$$

with appropriate initial conditions and a time-discretization at steps $\{0, 1, ..., i, ...\}$ with increment Δt, the discretization is

$$\begin{aligned} v_{i+\frac{1}{2}} &= v_i + \frac{1}{2}a(x_i, v_i, t_i)\Delta t \\ x_{i+1} &= x_i + \Delta t v_{i+\frac{1}{2}} \\ v_{i+1} &= v_i + \frac{\Delta t}{2}[a(x_i, v_i, t_i) + a(x_{i+1}, v_{i+\frac{1}{2}}, t_{i+1})]. \end{aligned} \tag{58}$$

5. NUMERICAL RESULTS AND HIGHER DIMENSIONS

In this section we present numerical simulations of one-dimensional nonlocal flocking dynamics, by solving (a) the agent-based Cucker-Smale model using the velocity Verlet method, and (b) the macroscopic model with initial conditions whose support is the interval $[-\pi, \pi]$. Our aim is to verify that the agent based and continuum based approaches to the flocking problem produce similar results.

In the following, the initial density and velocity are given by

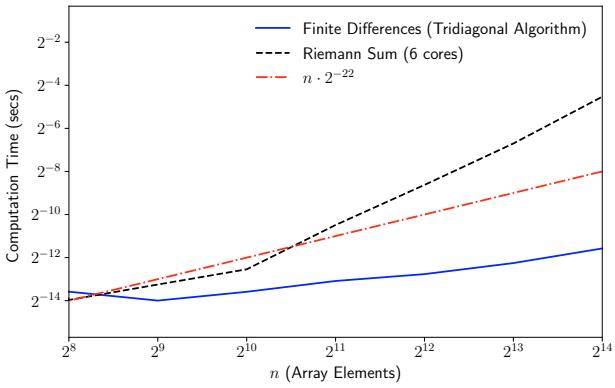

Fig. 2. Computation Times for Nonlocal Terms using Finite Differences and Riemann Sum.

$$\rho_0(\hat{x}) = \frac{\pi}{2L}\cos(\frac{\pi\hat{x}}{L}), \tag{59}$$

$$u_0(\hat{x}) = -c\sin(\frac{\pi\hat{x}}{L}),\ \hat{x} \in [-\frac{L}{2}, \frac{L}{2}], \tag{60}$$

i.e. it is assumed that $\rho_0(\hat{x}) = u_0(\hat{x}) = 0,\ \forall\hat{x} \notin [-\frac{L}{2}, \frac{L}{2}]$, where we have used $L = 2\pi$.

5.1 Cucker-Smale Model Simulation

In all simulations, we take $\lambda = 1$, $k = 4$. For the particle simulation, we use $N = 10^4$ particles. For the macro-scale simulation, we use $\Delta\hat{x} = \frac{2\pi}{600}$ as the spatial increment. In both simulations, we take $\Delta t = .001$ as the time increment.

In both cases, the support of the initial profile shrinks as the bulk comes together. The semi-linear Poisson-forced Euler system is highly dissipative, and the momentum profile is damped until it flattens (although it is conserved over the domain), and the system attains an equilibrium distribution. Fig. 3 shows the agreement between the particle model and the macro-scale model.

5.2 Higher Dimensions

In higher dimensions, the radial symmetry of the interaction function ψ suggests the use of a *singular kernel*. Singular kernels have been extensively studied in the literature and, under mild assumptions in the initial conditions, have been shown to result in flocking behavior while, at the same time, avoiding collisions (Ahn et al., 2012).

In the BVP of the augmented system of PDEs (21) with the initial and boundary conditions (26), we select the linear differential operator (see also (Mavridis et al., 2020)):

$$\mathcal{L}_x = -k^{-d/2}(\nabla^2_x - \lambda^2) \tag{61}$$

and $\Omega = B_d(0,r) := \{x \in \mathbb{R}^d : \|x\| < r\}$, which results in a Green's function of the form

$$\hat{\psi}(x,s) = \psi(x-s) + \phi(x,s) \tag{62}$$

where ψ is given by

$$\begin{aligned} \psi(x,s) &= \tilde{\psi}(\|x-s\|) \\ &= \left(\frac{k}{2\pi}\right)^{d/2}\left(\frac{\lambda}{\|x-s\|}\right)^{d/2-1} K_{d/2-1}(\lambda\|x-s\|) \end{aligned} \tag{63}$$

with $K_\alpha(\cdot)$ being the modified Bessel function of the second kind of order α, and ϕ is a function such that

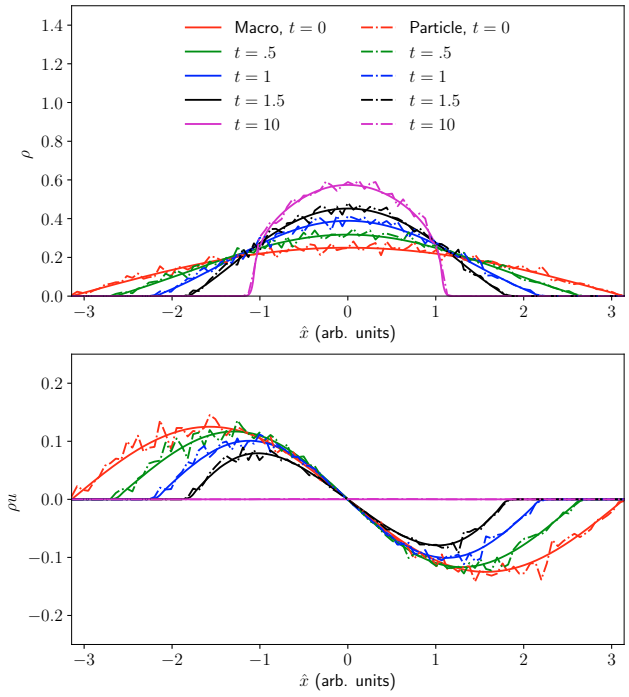

Fig. 3. Evolution of the Probability Densities $\rho(t,\hat{x})$ and Momentum Densities $m(t,\hat{x})$ as computed by solving the macroscale model and the particle model (dashed-line).

$$\begin{aligned} &\mathcal{L}_s\phi(x,s)=0,\ s\in B_d(0,r)\\ &\phi(x,s)=-\psi(x,s),\ s\in\partial B_d(0,r) \end{aligned} \tag{64}$$

For $s\in\partial B_d(0,r)$ we have

$$\begin{aligned} \|x-s\|^2 &= \|x\|^2-2\langle x,s\rangle+\|s\|^2\\ &= \|x\|^2\|\frac{s}{r}-\frac{rx}{\|x\|^2}\|^2 \end{aligned} \tag{65}$$

and it can be shown that

$$\phi(x,s)=-\tilde{\psi}(\frac{1}{r}\|x\|\|s-r^2\frac{x}{\|x\|^2}\|). \tag{66}$$

The interaction function $\hat{\psi}$ is affected by the bounded domain in the same way as in the one-dimensional case, and depends on the parameter values k and λ as illustrated in Fig.4 for the 2-dimensional case.

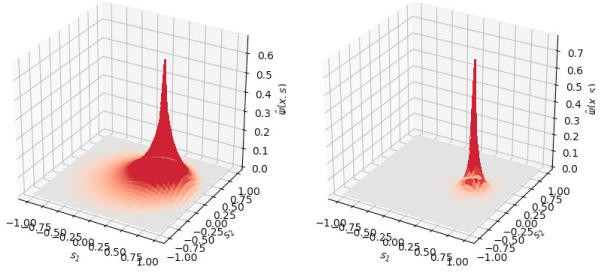

Fig. 4. The effect of the parameters k, λ on the profile of the interaction function $\hat{\psi}((0,0.5),s)$, $s\in B_2(0,1)$. Left: $(k,\lambda)=(1,0.5)$. Right: $(k,\lambda)=(2,10)$.

6. CONCLUSION

A family of compactly supported parametric interaction functions in the general Cucker-Smale flocking dynamics was proposed such that the macroscopic system of mass and momentum balance equations with non-local damping terms can be converted to an augmented system of coupled PDEs in a compact set. We approached the computation of the non-local damping using the standard finite difference treatment of the chosen differential operator, which was solved using banded matrix algorithms. The expressiveness of the proposed interaction functions may be utilized for parametric learning from trajectory data.

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